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Advanced Linear Algebra (MA 409)
Problem Sheet - 14

Systems of Linear Equations - Computational Aspects

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1. Label the following statements as true or false.

   (a) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary column operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

   (b) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary row operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

   (c) If A is an $n \times n$ matrix with rank n, then the reduced row echelon form of A is I_n.

   (d) Any matrix can be put in reduced row echelon form by means of a finite sequence of elementary row operations.

   (e) If $(A|b)$ is in reduced row echelon form, then the system $Ax = b$ is consistent.

   (f) Let $Ax = b$ be a system of m linear equations in n unknowns for which the augmented matrix is in reduced row echelon form. If this system is consistent, then the dimension of the solution set of $Ax = 0$ is $n - r$, where r equals the number of nonzero rows in A.

   (g) If a matrix A is transformed by elementary row operations into a matrix A' in reduced row echelon form, then the number of nonzero rows in A' equals the rank of A.

2. Use Gaussian elimination to solve the following systems of linear equations.

   a) $x_1 + 2x_2 - x_3 = -1$
   $2x_1 + 2x_2 + x_3 = 1$
   $3x_1 + 5x_2 - 2x_3 = -1$

   b) $x_1 - 2x_2 - x_3 = 1$
   $2x_1 - 3x_2 + x_3 = 6$
   $3x_1 - 5x_2 = 7$
   $x_1 + 5x_3 = 9$

   c) $x_1 + 2x_2 + 2x_4 = 6$
   $3x_1 + 5x_2 - x_3 + 6x_4 = 17$
   $2x_1 + 4x_2 + x_3 + 2x_4 = 12$
   $2x_1 - 7x_3 + 11x_4 = 7$

   d) $x_1 - x_2 - 2x_3 + 3x_4 = -7$
   $2x_1 - x_2 + 6x_3 + 6x_4 = -2$
   $-2x_1 + x_2 - 4x_3 - 3x_4 = 0$
   $3x_1 - 2x_2 + 9x_3 + 10x_4 = -5$

   e) $x_1 - 4x_2 - x_3 + x_4 = 3$
   $2x_1 - 8x_2 + x_3 - 4x_4 = 9$
   $-x_1 + 4x_2 - 2x_3 + 5x_4 = -6$

   f) $x_1 + 2x_2 - x_3 + 3x_4 = 2$
   $2x_1 + 4x_2 - x_3 + 6x_4 = 5$
   $x_2 + 2x_4 = 3$

   g) $2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1$
   $x_1 - x_2 + x_3 + 2x_4 - x_5 = 2$
   $4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6$

   h) $3x_1 - x_2 + x_3 - x_4 + 2x_5 = 5$
   $x_1 - x_2 - x_3 - 2x_4 - x_5 = 2$
   $5x_1 - 2x_2 + x_3 - 3x_4 + 3x_5 = 10$
   $2x_1 - x_2 - 2x_4 + x_5 = 5$

   i) $3x_1 - x_2 + 2x_3 + 4x_4 + x_5 = 2$
   $x_1 - x_2 + 2x_3 + 3x_4 + x_5 = -1$
   $2x_1 - 3x_2 + 6x_3 + 9x_4 + 4x_5 = -5$
   $7x_1 - 2x_2 + 4x_3 + 8x_4 + x_5 = 6$

   j) $2x_1 + 3x_3 - 4x_5 = 5$
   $3x_1 - 4x_2 + 8x_3 + 3x_4 = 8$
   $x_1 - x_2 + 2x_3 + x_4 - x_5 = 2$
   $-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 = -8$

3. Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $(A'|b')$ in reduced row echelon form by a finite sequence of elementary row operations.

   (a) Prove that $rank(A') \neq rank(A'|b')$ if and only if $(A'|b')$ contains a row in which the only nonzero entry lies in the last column.

   (b) Deduce that $Ax = b$ is consistent if and only if $(A'|b')$ contains no row in which the only nonzero entry lies in the last column.

4. For each of the systems that follow, apply Exercise 3 to determine whether the system is consistent. If the system is consistent, find all solutions. Finally, find a basis for the solution set of the corresponding homogeneous system.

   a) $x_1 + 2x_2 - x_3 + x_4 = 2$
   $2x_1 + x_2 + x_3 - x_4 = 3$
   $x_1 + 2x_2 - 3x_3 + 2x_4 = 2$

   b) $x_1 + x_2 - 3x_3 + x_4 = -2$
   $x_1 + x_2 + x_3 - x_4 = 2$
   $x_1 + x_2 - x_3 = 0$

   c) $x_1 + x_2 - 3x_3 + x_4 = 1$
   $x_1 + x_2 + x_3 - x_4 = 2$
   $x_1 + x_2 - x_3 = 0$

5. Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

   Determine A if the first, second, and fourth columns of A are

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

   respectively.

6. Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   Determine A if the first, third, and sixth columns of A are

$$\begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 2 \\ -4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 3 \\ -9 \\ 2 \\ 5 \end{pmatrix},$$

   respectively.

7. It can be shown that the vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

8. Let W denote the subspace of $\mathbb{R}^5$ consisting of all vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2,-3,4,-5,2), & u_2 &= (-6,9,-12,15,-6), \\ u_3 &= (3,-2,7,-9,1), & u_4 &= (2,-8,2,-2,6), \\ u_5 &= (-1,1,2,1,-3), & u_6 &= (0,-3,-18,9,12), \\ u_7 &= (1,0,-2,3,-2), \quad \text{and} & u_8 &= (2,-1,1,-9,7) \end{aligned}$$

generate W. Find a subset of $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

9. Let W be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of the symmetric 2×2 matrices. The set

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

generates W. Find a subset of S that is a basis for W.

10. Let

$$V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}.$$

(a) Show that $S = \{(0,1,1,1,0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

11. Let V be as in Exercise 10.

(a) Show that $S = \{(1,2,1,0,0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

12. Let V denote the set of all solutions to the system of linear equations

$$\begin{aligned} x_1 - x_2 + 2x_4 - 3x_5 + x_6 &= 0 \\ 2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 &= 0. \end{aligned}$$

(a) Show that $S = \{(0,-1,0,1,1,0), (1,0,1,1,1,0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

13. Let V be as in Exercise 12.

(a) Show that $S = \{(1,0,1,1,1,0), (0,2,1,1,0,0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

14. If $(A|b)$ is in reduced row echelon form, prove that A is also in reduced row echelon form.

15. Prove that the reduced row echelon form of a matrix is unique.

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